DIFFRACTION OF PLANE ELASTIC WAVES BY SMOOTH CONVEX CYLINDERS

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The diffraction of plane elastic waves by smooth convex cylinders is studied. The diffraction of longitudinal and transverse plane waves by a circular stress-free cylinder is considered in Sections 1-7. The solution of similar problems is given in series form in [1]. However, rapid convergence of these series only occurs in the case when the diameter of the cylinder a is small compared with the wavelength $a \ll 2\pi/k$. We shall concern ourselves with the high-frequency case, i.e., when $k_1 a \gg 1$ and $k_2 a \gg 1$. In [2,3] a quite general method of solution is proposed for the problem of scattering of high-frequency elastic waves by curved surfaces. However, the phenomena which occur are studied (in accordance with Kirchhoff's principle) only in the 'illuminated' region. The diffraction of longitudinal elastic waves emanating from a line source by a rigid cylinder is studied in [4] using a double Laplace transformation in time and angle. The solution is found in the illuminated region and in the shadow for the first arrival of the waves.

In this paper a method is used which was first proposed by Watson [5] and later developed by V.A. Fok [6] and other authors [7-10], etc. for problems in accoustics and electrodynamics. This uses the transformation of the sums of series into contour integrals (Section 2). Then the elastic displacements outside the cylinder, which are expressed in terms of complex integrals, are investigated in various regions of the elastic space (Sections 3-6). Short-wave asymptotic expansions for the displacements are found. The diffraction of a longitudinal elastic wave by a cylindrical cavity is considered in Section 7. In Section 8 the diffraction of plane waves by smooth convex cylinders of arbitrary section is studied with the aid of the 'geometrical theory of diffraction' proposed by Keller and his co-authors [11, 12].

1. Formulation of the problem. A plane transverse wave having the potential $\psi_0 = \exp(ik_2 r \cos\vartheta - i\omega t)$ is incident on a cylindrical surface of radius a in an infinite elastic space (Fig. 1). The problem consists of studying the displacements generated in the elastic space outside the cylinder by the diffraction of the transverse wave We shall consider the cylindrical surface to be stress-free i.e., $\sigma_{rr} = 0$, $\tau_{r\vartheta} = 0$ for $r = a$. It is well known that the potentials of longitudinal and transverse waves in an elastic medium satisfy the Helmholtz equations

$$\Delta\varphi + k_1^2\varphi = 0, \qquad \Delta\psi + k_2^2\psi = 0 \tag{1.1}$$

where k_1 and k_2 are the wave numbers of the longitudinal and transverse waves. The solution of Eqs. (1.1) together with the boundary conditions indicated has the form [1].

$$\varphi_1 = \frac{4i}{\pi}\sum_0^\infty \varepsilon_n (i)^n \frac{A(n)}{\Delta_n} H_n^{(1)}(k_1 r)\sin n\vartheta$$

$$\psi_0 + \psi_1 = \sum_0^\infty \varepsilon_n (i)^n \left[-\frac{\Delta_{n1}}{\Delta_n} H_n^{(1)}(k_2 r) + J_n(k_2 r)\right]\cos n\vartheta$$

where $H_n^{(1)}(\rho)$ are Hankel functions of the first kind, $J_n(\rho)$ are Bessel functions.

$$\varepsilon_0 = 1, \qquad \varepsilon_n = 2 \qquad (n = 1, 2, \ldots), \qquad x = k_1 a, \qquad y = k_2 a$$

$$\Delta_n = [4n^2 - (2n^2 - y^2)^2]\, H_n^{(1)}(x)\, H_n^{(1)}(y) + 4xy\,(n^2 - 1)\, H_n^{(1)\prime}(x)\, H_n^{(1)\prime}(y) -$$
$$- 2y^2\,[xH_n^{(1)\prime}(x)\, H_n^{(1)}(y) + yH_n^{(1)\prime}(y)\, H_n^{(1)}(x)], \quad A(n) = n\,(2n^2 - 2 - y^2)$$

The function Δ_{n1} is analogous to Δ_n with $J_n(y)$ replacing $H_n^{(1)}(y)$. The displacement field outside the cylinder may be written as follows:

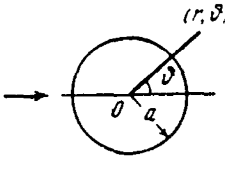

FIG. 1.

$$u = \frac{\partial\varphi_1}{\partial r} + \frac{1}{r}\frac{\partial(\psi_0 + \psi_1)}{\partial\theta} =$$

$$= \frac{4ik_1}{\pi}\sum_0^\infty \varepsilon_n \frac{A(n)}{\Delta_n} H_n^{(1)\prime}(k_1 r)\sin n\vartheta e^{in\pi/2} + \tag{1.2}$$

$$+ \frac{1}{r}\sum_0^\infty \varepsilon_n n\left[\frac{\Delta_{n1}}{\Delta_n} H_n^{(1)\prime}(k_2 r) - J_n(k_2 r)\right]\sin n\vartheta e^{in\pi/2}$$

Here and later on we shall consider expressions for just the radial component u of the displacement field and we shall give the expressions for the tangential component v only in the final results. The first sum in (1.2) represents the displacement in the longitudinal (p) waves and the second the displacements in the transverse (s) waves. We denote them by u^p and u^s, respectively.

2. Transformation of the displacements into the form of contour integrals. If $k_1 a \gg 1$ and $k_2 a \gg 1$, the series of (1.2) converge very slowly. We shall, therefore, transform the sums of the series in (1.2) into integrals along the path C in the complex plane ν (Fig. 2) by Watson's method [5]

$$u^p = -\frac{4k_1}{\pi}\int_C \frac{A(\nu)}{\Delta_\nu} H_\nu^{(1)'}(k_1 r)\frac{\sin\nu\vartheta}{\sin\nu\pi} e^{-i\nu\pi/2}\,d\nu + u_R{}^p \equiv u_1{}^p + u_R{}^p$$

$$u^s = \frac{i}{r}\int_C \nu\left[\frac{\Delta_{\nu 1}}{\Delta_\nu} H_\nu^{(1)}(k_2 r) - J_\nu(k_2 r)\right]\frac{\sin\nu\vartheta}{\sin\nu\pi} e^{-i\nu\pi/2}\,d\nu + \qquad (2.1)$$

$$u_R{}^s \equiv u_1^s + u_R{}^s$$

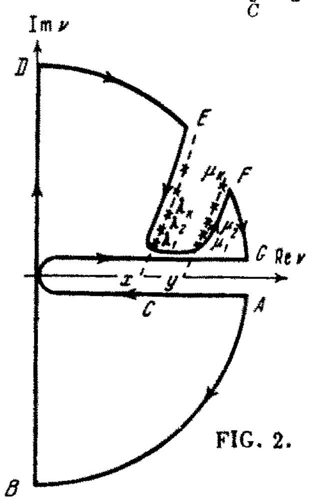

FIG. 2.

where $u_R{}^p$ and $u_R{}^s$ are the residues of the integrands at the real zero $\nu^\times$ of the equation $\Delta_\nu = 0$.

As will be shown in Section 3, $\Delta_\nu = 0$ has, in addition to $\nu^\times$, two sequences of complex roots $\lambda_1, \ldots, \lambda_k, \ldots$ and $\mu_1, \ldots, \mu_k, \ldots$, which lie in the first quadrant. Therefore, the integrals along the contour C can be replaced by integrals along the path $ABDEFG$, where A, B, D, ... are points on a circle of infinitely large radius, and the path EF envelops the sequences of roots λ_k and μ_k (Fig. 2). It is easy to show that the integrands of (2.1) are such that all the integrals over the parts of the infinite circle go to zero. Each integral of (2.1) along C can then be represented in the form of the sum of two integrals along BD and EF. All the integrals along BD go to zero as a consequence of the oddness of the integrands. We transform (2.1) using the Wronskian of the Bessel functions

$$u_1{}^p = -\frac{4k_1}{\pi}\int_E^F \frac{A(\nu)}{\Delta_\nu} H_\nu^{(1)'}(k_1 r)\frac{\sin\nu\vartheta}{\sin\nu\pi} e^{-i\nu\pi/2}\,d\nu \qquad (2.2)$$

$$u_1{}^s = \frac{i}{r}\int_E^F \nu\left\{\left[\frac{J_\nu(y)}{H_\nu^{(1)}(y)} H_\nu^{(1)}(k_2 r) - J_\nu(k_2 r)\right] + \frac{4i}{\pi}\frac{B(\nu)}{\Delta_\nu}\frac{H_\nu^{(1)}(k_2 r)}{H_\nu^{(1)}(y)}\right\}\frac{\sin\nu\vartheta}{\sin\nu\pi} e^{-i\nu\pi/2}\,d\nu$$

$$(B(\nu) = 2x(1-\nu^2)H_\nu^{(1)'}(x) + y^2 H_\nu^{(1)}(x))$$

The integrals in (2.2) can be computed in terms of the residues at the zeros of $\Delta_\nu = 0$

$$u_1{}^p = -8ik_1\sum_k \frac{A(\nu_k)}{(\partial\Delta_\nu/\partial\nu)_{\nu_k}} H_{\nu_k}{}^{(1)'}(k_1 r)\frac{\sin\nu_k\vartheta}{\sin\nu_k\pi}\exp\frac{-i\nu_k\pi}{2} \qquad (2.3)$$

$$u_1{}^s = -\frac{8i}{r}\sum_k \nu_k\frac{B(\nu_k)}{(\partial\Delta_\nu/\partial\nu)_{\nu_k}}\frac{H_{\nu_k}{}^{(1)}(k_2 r)}{H_{\nu k}{}^{(1)}(y)}\frac{\sin\nu_k\vartheta}{\sin\nu_k\pi}\exp\frac{-i\nu_k\pi}{2} \qquad (2.4)$$

where, for brevity, we denote both sets of complex zeros λ_k and μ_k of the function Δ_ν by $\nu_1, \ldots, \nu_k \ldots$

The convergence of the series (2.3) and (2.4) follows from the convergence of the series (1.2). However, for practical application of the series (2.3) and (2.4) rapid convergence beginning with the first few terms of the series is required. As we shall see later, the series (2.3) and (2.4) converge well and are useful for application in limited regions of the space. In order to extend the solution to the remaining regions, we shall transform the integrals (2.2), by making use of the relations

$$\frac{\sin\nu\vartheta}{\sin\nu\pi} = -\frac{\sin\nu(\pi-\vartheta)}{\sin\nu\pi}e^{i\nu\pi} + e^{i\nu(\pi-\vartheta)},\quad \frac{\cos\nu\vartheta}{\sin\nu\pi} = \frac{\cos\nu(\pi-\vartheta)}{\sin\nu\pi}e^{i\nu\pi} - ie^{i\nu(\pi-\vartheta)} \tag{2.5}$$

Each of the displacements in (2.2) then consists of the sum of two integrals along EF. The first integrals, just as before, are calculated by means of the residues at the zeros of Δ_ν; the second integrals may be evaluated by the method of steepest descent, in which case the additional terms u_{1R}^{p}, and u_{1R}^{s}, the residues of the integrand at the zero $\nu = \nu^{\times}$, will appear. Thus

$$u^p = 8ik_1\sum_k \frac{A(\nu_k)}{(\partial\Delta_\nu/\partial\nu)_{\nu_k}} H_{\nu_k}^{(1)\prime}(k_1 r)\frac{\sin\nu_k(\pi-\vartheta)}{\sin\nu_k\pi}\exp\frac{i\nu_k\pi}{2} + \tag{2.6}$$

$$+\frac{4k_1}{\pi}\int\frac{A(\nu)}{\Delta_\nu}H_\nu^{(1)\prime}(k_1 r)\,e^{i\nu(1/2\pi-\vartheta)}\,d\nu + u_R^{p} + u_{1R} \equiv u_2^{p} + U^{p} + u_{2R}^{p}$$

$$u^s = \frac{8i}{r}\sum \nu_k\frac{B(\nu_k)}{(\partial\Delta_\nu/\partial\nu)_{\nu_k}}\frac{H_{\nu_k}^{(1)}(k_2 r)}{H_{\nu_k}^{(1)}(y)}\frac{\sin\nu_k(\pi-\vartheta)}{\sin\nu_k\pi}\exp\frac{i\nu_k\pi}{2} -$$

$$-\frac{i}{r}\int\nu\left[\frac{H_\nu^{(2)}(y)}{2} + \frac{4i}{\pi}\frac{B(\nu)}{\Delta_\nu}\right]\frac{H_\nu^{(1)}(k_2 r)}{H_\nu^{(1)}(y)}e^{i\nu(1/2\pi-\vartheta)}\,d\nu + \tag{2.7}$$

$$+ u_R^{s} + u_{1R}^{s} \equiv u_2^{s} + U^{s} + u_0 + u_{2R}^{s}$$

In the Eqs. (2.6) and (2.7), the series in ν_k for u_2^P and u_2^s converge rapidly everywhere; the integrals along the suitable steepest descent paths are denoted by U^P and $U^s + u_0$.

3. The roots of the equation $\Delta_\nu = 0$ for $k_1 a \gg 1$ and $k_2 a \gg 1$. Surface waves. All the arguments presented thus far have been exact. Later we shall study the short-wave asymptotic expressions for the elastic displacements, $k_1 a \gg 1$, and $k_2 a \gg 1$. We shall, therefore, replace the Bessel functions occurring in the solution by their asymptotic representations.

In the regions of the complex plane ν where

$$|\nu^2 - x^2| \gg Ax^{4/3},\qquad |\nu^2 - y^2| \gg Ay^{4/3},\qquad A \sim 2.5 \tag{3.1}$$

(we shall call these the Debye regions), the asymptotic expressions of Debye [13, 14] are valid for Bessel functions with arguments x and y. The more complicated Hankel-Fok asymptotic expressions [6]

$$H_\nu^{(1)}(\rho) = \frac{-i}{\sqrt{\pi}}\left(\frac{\rho}{2}\right)^{-1/3} w(t),\qquad J_\nu(\rho) = \frac{1}{\sqrt{\pi}}\left(\frac{\rho}{2}\right)^{-1/3} v(t)$$

$$H_\nu^{(2)}(\rho) = \frac{i}{\sqrt{\pi}}\left(\frac{\rho}{2}\right)^{-1/3} w(e^{2/3\pi i}t),\qquad t = (\nu-\rho)\left(\frac{\rho}{2}\right)^{-1/3} \tag{3.2}$$

must be used for the Bessel functions in the regions where the inequalities (3.1) are not satisfied (we shall call these the Fok regions). Here $w(t) = u(t) + iv(t)$ is the Airy function [6].

For $x \gg 1$ and $y \gg 1$, the roots of $\Delta_\nu = 0$ are of the order of these quantities. Therefore, dropping terms of higher order, we obtain

$$\Delta_\nu \approx -(2\nu^2 - y^2)^2 H_\nu^{(1)}(x) H_\nu^{(1)}(y) + 4xy\nu^2 H_\nu^{(1)\prime}(x) H_\nu^{(1)\prime}(y) = 0 \tag{3.3}$$

Outside the Fok regions near the real axis this expressions assumes the form

$$\Delta_\nu \approx H_\nu^{(1)}(x) H_\nu^{(1)}(y) \left[-(2\nu^2 - y^2)^2 + 4\nu^2 \sqrt{\nu^2 - x^2}\sqrt{\nu^2 - y^2}\right] = 0$$

The expression in brackets is Rayleigh's equation, which has a real positive root $\nu^\times = \kappa y$, where $\kappa > 1$ depends on $\epsilon = x/y < 1$. The other roots of Eq. (3.3), λ_k and μ_k are located in the first quadrant near the lines of zeros of $H_\nu^{(1)}(x)$ and $H_\nu^{(1)}(y)$ which pass through the points $y = x$ and $\nu = y$, respectively. In a Fok region of the function $H_\nu^{(1)}(x)$, the following expression is valid:

$$|\nu^2 - x^2| \sim Ax^{4/3} \quad \text{or} \quad |t| = (1/2x)^{1/3}|\nu - x| \sim 1$$

Therefore

$$\Delta_\nu \approx -4x^3 y \sqrt{1-\varepsilon^2}\,\pi^{-1/2}(1/2x)^{-2/3}[w'(t) - q_1 w(t)] H_\nu^{(1)}(y) = 0 \tag{3.4}$$
$$(q_1 = i(2\varepsilon^2 - 1)^2 4^{-1}\varepsilon^{-3}(1-\varepsilon^2)^{-1/2}(1/2x)^{1/3})$$

For $q_1 = 0$ the roots of (3.4) coincide with t_k', the zeros of $w'(t)$, and for $q_1 = \infty$ they are t_k^0, the zeros of $w(t)$. Equations of the type (3.4) are studied in detail in [6,7] and series for the computations of t_k, the roots of (3.4) for $|q_1/\sqrt{t_k}| > 1$ and $|q_1/\sqrt{t_k}| < 1$. are given there. For $|q_1/\sqrt{t_k}| \sim 1$ it is necessary to use the equation $dt/dq_1 = 1/(t - q_1^2)$, which is a consequence of (3.4) and the definition of the Airy function $w(t)$, and to solve it numerically with the initial conditions $t_k = t_k'$ for $q_1 = 0$ or $t_k = t_k^\circ$ for $q_1 = \infty$. The roots of Eq. (3.4) are located in the first quadrant near the straight line $\arc t = 1/3\pi$.

In a Fok region of the function $H_\nu^{(1)}(y)$

$$\Delta_\nu = -4iy^4\sqrt{1-\varepsilon^2}\,\pi^{-1/2}(1/2y)^{-2/3}[w'(\tau) - q_2 w(\tau)] H_\nu^{(1)}(x) = 0$$
$$q_2 = 4^{-1}(1-\varepsilon^2)^{-1/2}(1/2y)^{1/3}, \qquad \tau = (\nu - y)(1/2y)^{-1/3} \tag{3.5}$$

The roots r_k of Eq. (3.5) lie near the straight line $\arc \tau = 1/3\pi$ in the first quadrant.

Using the above asymptotic expressions, we shall now give an interpretation of the terms with the index R in (2.1). Mathematically, they are computed as the residues of the integrands at the pole $\nu^\times = \kappa y$. The expressions for these displacements on the surface of the cylinder assume the form

$$(u_R^p)_a = 2\sqrt{2\pi y}k_2\varkappa^2(2\varkappa^2-1)^3\Lambda^2(\varepsilon,a)\Lambda(1,a)\Delta_0^{-1}e^{y\Theta}\sin\varkappa y\vartheta$$
$$(u_R^s)_a = -4\sqrt{2\pi y}k_2\varkappa^4(2\varkappa^2-1)^2\Lambda^2(\varepsilon,a)\Lambda(1,a)\Delta_0^{-1}e^{y\Theta}\sin\varkappa y\vartheta$$
$$(v_R^p)_a = -2\sqrt{2\pi y}k_2\varkappa^3(2\varkappa^2-1)^3\Lambda(1,a)\Delta_0^{-1}e^{y\Theta}\cos\varkappa y\vartheta$$
$$(v_R^s)_a = \sqrt{2\pi y}k_2\varkappa(2\varkappa^2-1)^4\Lambda(1,a)\Delta_0^{-1}e^{y\Theta}\cos\varkappa y\vartheta$$
$$\Delta_0 = [8\varkappa^6(\varepsilon^2-1)+4\varkappa^2-1]\sin\varkappa y\pi$$
$$\Lambda(\varepsilon,r) = \sqrt[4]{\varkappa^2a^2/r^2-\varepsilon^2}, \qquad \Theta = \sqrt{\varkappa^2-1}-\varkappa\,|\arc\cos\varkappa\,|-{}^1/_2 i\varkappa\pi$$

which are correct up to terms of order y^{-1}. And near the surface of the cylinder

$$u_R^p \approx (u_R^p)_a (a/r)^{1/2}\Lambda(\varepsilon,r)\Lambda^{-1}(\varepsilon,a)\exp[-(r-a)k_2\Lambda^2(\varepsilon,a)]$$
$$v_R^p \approx (v_R^p)_a (a/r)^{3/2}\Lambda(\varepsilon,a)\Lambda^{-1}(\varepsilon,r)\exp[-(r-a)k_2\Lambda^2(\varepsilon,a)]$$
$$u_R^s \approx (u_R^s)_a (a/r)^{3/2}\Lambda(1,a)\Lambda^{-1}(1,r)\exp[-(r-a)k_2\Lambda^2(\varepsilon,1)]$$
$$v_R^s \approx (v_R^r)_a (a/r)^{1/2}\Lambda(1,r)\Lambda^{-1}(1,a)\exp[-(r-a)k_2\Lambda^2(\varepsilon,1)]$$

Thus, the displacements having index R propagate along the surface of the cylinder with the velocity $\pm b/\kappa$ and decay exponentially with distance away from the cylinder; i.e., they correspond to the motions of Rayleigh surface waves.

4. The geometrical portion of the displacement field. Let us study the physical meaning of the components of the displacements which are expressed in (2.6) and (2.7) by integrals along paths of steepest descent. In the region where (3.1) is satisfied, $k_1r - x \gg x^{1/3}$ and $k_2r-y \gg y^{1/3}$ let us apply the Debye asymptotic expressions for the Bessel functions. We note that the path of steepest descent for the first term in the integral of (2.7) must pass through the two saddle points ν_0 and ν_{00} $(\nu_0 < y < \nu_{00})$ and through the first zero of the function $H_\nu^{(2)}(y)$.

$$U^p = \frac{2\sqrt{2}}{r\sqrt{\pi}}e^{-i\pi/4}\int\frac{\nu(2\nu^2-y^2)[(k_1^2r^2-\nu^2)(x^2-\nu^2)(y^2-\nu^2)]^{1/4}}{\Delta^+(\nu)}e^{i\Phi_1}d\nu \tag{4.1}$$

$$U^s + u_0 = \frac{e^{3i\pi/4}}{r\sqrt{2\pi}}\int\frac{\Delta^-(\nu)}{\Delta^+(\nu)\sqrt[4]{k_2^2r^2-\nu^2}}e^{i\Phi_2}d\nu - \frac{e^{3i\pi/4}}{r\sqrt{2\pi}}\int\frac{\nu}{\sqrt[4]{k_2^2r^2-\nu^2}}e^{i\Phi_3}d\nu \tag{4.2}$$

$$\Phi_1(\nu) = \sqrt{k_1^2r^2-\nu^2}-\sqrt{x^2-\nu^2}-\sqrt{y^2-\nu^2}-$$
$$-\nu\left(\arc\cos\frac{\nu}{k_1r}-\arc\cos\frac{\nu}{x}-\arc\cos\frac{\nu}{y}-{}^1/_2\pi+\vartheta\right)$$
$$\Phi_2(\nu) = \sqrt{k_2^2r^2-\nu^2}-2\sqrt{y^2-\nu^2}-\nu\left(\arc\cos\frac{\nu}{k_2r}-2\arc\cos\frac{\nu}{y}-{}^1/_2\pi+\vartheta\right)$$
$$\Phi_3(\nu) = \sqrt{k_2^2r^2-\nu^2}-\nu(\arc\cos\nu/k_2r-{}^1/_2\pi+\vartheta) \tag{4.3}$$
$$\Delta^\pm(\nu) = 4\nu^2\sqrt{x^2-\nu^2}\sqrt{y^2-\nu^2}\pm(2\nu^2-y^2)^2$$

In (4.2) the first integral is taken along a contour passing through the left saddle ν_0, while the second is along a path through the right one ν_{00} and gives the displacement field in the incident wave u_0.

The saddle point ν^0 for the integrals of (4.1) is determined, according to (4.3), by the equation

$$\Phi_1'(\nu^\circ) = -\arccos(\nu^\circ / k_1 r) + \arccos(\nu^\circ / x) + \arccos(\nu^\circ / y) + {}^1/_2\pi - \vartheta = 0 \tag{4.4}$$

Let us set $\nu^\circ / y = \sin\alpha_1$, $\nu^\circ / x = \sin\beta_1$, $\nu^\circ / k_1 r = \sin\delta_1$, Eq. (4.4) will then be satisfied if $\delta_1 = \alpha_1 + \beta_1 + \vartheta - \pi$.

The geometrical significance of the angles is shown in Fig. 3.

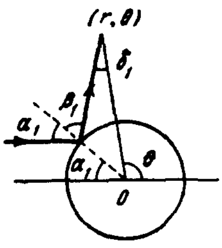

FIG. 3

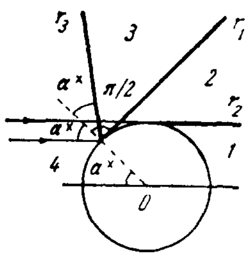

FIG. 4

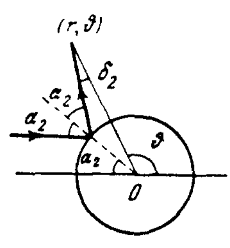

FIG. 5

The method of steepest descents is applicable to (4.1) as long as $|\nu^{\circ 2} - x^2| \gg A x^{4/3}$ and $\cos\beta_1 \gg x^{-1/3}$. The point $\nu^0 = x$ corresponds in the physical space to the angles $\beta_1 = {}^1/_2\pi$ and $\alpha_1 = \alpha^\times$ ($\alpha^\times$ is the angle of total internal reflection). Thus the steepest descent approximation ceases to be valid for the computation of longitudinal motions near the boundary of the shadow of the longitudinal waves (Fig. 4)

$$r_1 = a / \cos(\vartheta^\times - \vartheta) \qquad (\vartheta^\times = \pi - \alpha^\times) \tag{4.5}$$

The saddle point ν_0 for (4.2) is determined by the condition

$$\Phi_2'(\nu_0) = -\arccos(\nu_0 / k_1 r) + 2\arccos(\nu_0 / y) + {}^1/_2\pi - \vartheta = 0 \tag{4.6}$$

We set $\nu_0 / y = \sin\alpha_2$ and $\nu_0 / k_1 r = \sin\delta_2$ (Fig. 5). Then (4.6) is satisfied if $\delta_2 = 2\alpha_2 + \vartheta - \pi$. The method of steepest descents can be applied formally to the integrals of (4.2) as long as $|\nu_0^2 - y^2| \gg A y^{4/3}$ and $\cos\alpha_2 \gg y^{-1/3}$, i.e., far from the boundary of the shadow of the transverse displacements

$$r_2 = a / \cos({}^1/_2\pi - \vartheta) \tag{4.7}$$

The neighborhood of the point $\nu_0 = x$ must also be excluded from the region of applicability of the method. In the physical space it corresponds to the region near the straight line

$$r_3 = a\varepsilon / \cos(2\alpha^\times + \vartheta + {}^1/_2\pi) \tag{4.8}$$

Moreover, it is necessary to bear in mind that when the path EF is deformed into a path of steepest descent in (2.7), some poles of the integrand will fall between the two paths. For the geometrically reflected waves we obtain the displacement field

$$U^p = -\frac{4ik_2a}{r}\,\frac{\cos\alpha_1\sin\alpha_1\,(2\sin^2\alpha_1-1)\,\Omega(\varepsilon,a,\alpha_1)\,\Omega(\varepsilon,r,\alpha_1)}{D^+(\alpha_1)\,W(\varepsilon,\alpha_1)}\times$$
$$\times\, e^{ik_2a\,[\Omega(\varepsilon,r,\alpha_1)-\Omega(\varepsilon,a,\alpha_1)-\cos\alpha_1]} \qquad (4.9)$$
$$V^p = \frac{4ik_2a}{r}\,\frac{\sin^2\alpha_1\cos\alpha_1\,(2\sin^2\alpha_1-1)\,\Omega(\varepsilon,a,\alpha_1)}{D^+(\alpha_1)\,W(\varepsilon,\alpha_1)}\times$$
$$\times\, e^{ik_2a\,[\Omega(\varepsilon,r,\alpha_1)-\Omega(\varepsilon,a,\alpha_1)-\cos\alpha_1]}$$

$$U^s = \frac{ik_2a}{r}\,\frac{\sin\alpha_2\cos\alpha_2 D^-(\alpha_2)}{D^+(\alpha_2)\,W(1,\alpha_2)}\,e^{ik_2a\,[\Omega(1,r,\alpha_2)-2\cos\alpha_2]}$$
$$V^s = -\frac{ik_2a}{r}\,\frac{\cos\alpha_2\,\Omega(1,r,\alpha_2)\,D^-(\alpha_2)}{D^+(\alpha_2)\,W(1,\alpha_2)}\,e^{ik_2a\,[\Omega(1,r,\alpha_2)-2\cos\alpha_2]} \qquad (4.10)$$

where

$$D^{\pm}(\alpha) = \Delta^{\pm}(y\sin\alpha)\,y^{-4}, \qquad \Omega(\varepsilon,r,\alpha) = \sqrt{\varepsilon^2r^2/a^2-\sin^2\alpha}$$
$$W(\varepsilon,\alpha) = \sqrt{\Omega(\varepsilon,r,\alpha)\,[\Omega(\varepsilon,a,\alpha)+\cos\alpha]-\cos\alpha\,\Omega(\varepsilon,a,\alpha)}$$

5. The diffracted displacement components. Each of the displacements in (2.2) and (2.4) is actually the sum of two series of residues at the poles λ_k and μ_k. For the practical application of these series they should converge so rapidly that it would be sufficient to take one or a few of the leading terms of the series. Since in the computation of the leading terms of the series of residues near x and y the poles lie in the Fok regions of the functions $H_\nu^{(1)}(x)$ or $H_\nu^{(1)}(y)$, the Hankel-Fok asymptotic expressions for these functions must be used. Then, instead of (2.3), we obtain

$$u_1^p = -L\left(\frac{a}{r}\right)^{1/2}\left(1-\frac{a^2}{r^2}\right)^{1/4}\sum_k\frac{\exp\,[i\lambda_k(\arccos\varepsilon-\arccos a/r-{}^1\!/_2\,\pi)]}{w(t_k)\,(t_k-q_1^2)}\,\frac{\sin\lambda_k\vartheta}{\sin\lambda_k\pi}$$
$$v_1^p \equiv iL\left(\frac{a}{r}\right)^{3/2}\left(1-\frac{a^2}{r^2}\right)^{-1/4}\sum_k\frac{\exp\,[i\lambda_k(\arccos\varepsilon-\arccos a/r-{}^1\!/_2\,\pi)]}{w(t_k)\,(t_k-q_1^2)}\,\frac{\cos\lambda_k\vartheta}{\sin\lambda_k\pi} \qquad (5.1)$$
$$L \equiv L(\varepsilon,k_1,k_2,a,r) =$$
$$= k_2(2\varepsilon^2-1)\,\pi^{1/2}\varepsilon^{-1/2}(1-\varepsilon^2)^{-1/4}\exp[i(k_1\sqrt{r^2-a^2}-k_2a\sqrt{1-\varepsilon^2})]$$
$$\lambda_k = x+({}^1\!/_2\,x)^{1/3}\,t_k$$

where the t_k are the roots of (3.4). The following asymptotic expression is valid near the straight line arc $t = {}^1\!/_3\pi$ [6]:

$$w(t_k) \approx \frac{2e^{1/4\,i\pi}(-1)^{k-1}}{\sqrt[4]{t_k}\sqrt{1-q_1^2/t_k^2}} \qquad (5.2)$$

Therefore, rapid convergence of the series (5.1) is determined by the factor

$$\exp\, i\, F_1 = \exp\, [i\lambda_k\, (\operatorname{arc\,cos} \varepsilon - \operatorname{arc\,cos}\, (a/r) - \vartheta + {}^1\!/_2\pi)]$$

and since the λ_k have positive imaginary parts, this condition

$$\operatorname{Im} F_1 > 0, \quad \text{or} \quad z_1 = \operatorname{arc\,cos} \varepsilon - \operatorname{arc\,cos}\, (a/r) - \vartheta + {}^1\!/_2\pi > 0 \qquad (5.3)$$

is satisfied to the right of the line (4.5) in Fig. 4. The series in μ_k in (2.3) results in purely surface disturbances. In an analogous fashion, we find that rapid convergence of the series in λ_k in (2.4) will occur whenever

$$z_2 = 2\, \operatorname{arc\,cos} \varepsilon - \operatorname{arc\,cos}\, (\varepsilon a/r) - \vartheta + {}^1\!/_2\pi > 0 \qquad (5.4)$$

i.e. in the region to the right of the line (4.8) in Fig. 4, and that the series for the transverse displacements themselves have the following asymptotic expressions:

$$(5.5)$$

$$u_{11}{}^s = iM \left(\frac{a}{r}\right)^{3/2} \left(1 - \frac{a^2}{r^2}\right)^{-1/4} \sum_k \frac{\exp\, [i\lambda_k\, (2\, \operatorname{arc\,cos} \varepsilon - \operatorname{arc\,cos} \varepsilon a/r - {}^1\!/_2\pi)]}{t_k - q_1{}^2} \, \frac{\sin \lambda_k \vartheta}{\sin \lambda_k \pi}$$

$$v_{11}{}^s = -M \left(\frac{a}{r}\right)^{1/2} \left(1 - \frac{a^2}{r^2}\right)^{1/4} \sum_k \frac{\exp\, [i\lambda_k\, (2\, \operatorname{arc\,cos} \varepsilon - \operatorname{arc\,cos} \varepsilon a/r - {}^1\!/_2\pi)]}{t_k - q_1{}^2} \, \frac{\cos \lambda_k \vartheta}{\sin \lambda_k \pi}$$

$$M \equiv M\, (\varepsilon, k_1, k_2, a, r) = k_1 8^{-1/2} \varepsilon^{-2} (1 - \varepsilon^2)^{-1/4} (2\varepsilon^2 - 1)^2 (\pi k_2 a)^{1/2} \times \\ \times (2/k_1 a)^{1/3} \exp\, [ik_2\, (\sqrt{r^2 - a^2\varepsilon^2} - 2a\sqrt{1 - \varepsilon^2}) + {}^1\!/_4\, i\pi]$$

Rapid convergence of the series in μ_k in (2.4) is determined by the condition

$$z_3 = - \operatorname{arc\,cos}\, (a/r) - \vartheta + {}^1\!/_2\pi > 0, \quad \text{or} \quad r < a / \cos\, ({}^1\!/_2\pi - \vartheta) \qquad (5.6)$$

This is the region of the geometric shadow *1* in Fig. 4. The asymptotic expressions for the series in μ_k of (2.4) are the following:

$$(5.7)$$

$$u_{12}{}^s = iN\, (k_2, r, a) \left(\frac{a}{r}\right)^{3/2} \left(1 - \frac{a^2}{r^2}\right)^{-1/4} \sum_k \frac{\exp\, [-i\mu_k\, (\operatorname{arc\,cos}\, a/r + {}^1\!/_2\pi)]}{w^2(\tau_k)\, (\tau_k - q_2{}^2)} \, \frac{\sin \mu_k \vartheta}{\sin \mu_k \pi}$$

$$v_{12}{}^s = -N\, (k_2, r, a) \left(\frac{a}{r}\right)^{1/2} \left(1 - \frac{a^2}{r^2}\right)^{1/4} \sum_k \frac{\exp\, [-i\mu_k\, (\operatorname{arc\,cos}\, a/r + {}^1\!/_2\pi)]}{w^2(\tau_k)\, (\tau_k - q_2{}^2)} \, \frac{\cos \mu_k \vartheta}{\sin \mu_k \pi}$$

$$N\, (k, r, a) = (8\pi k)^{1/2} a^{-1/2} ({}^1\!/_2\, ka)^{1/3} \exp\, [i\, (k\sqrt{r^2 - a^2} - {}^1\!/_4\pi)]$$

where τ_k are the roots of Eq. (3.5), and $\mu_k = y + ({}^1\!/_2\, y)^{1/3}\, \tau_k$.

The series (5.5) and (5.7) show that a transverse wave incident on the body produces two types of transverse diffracted waves, the physical significance of which will be clarified in Section 8.

Each of the series of (5.1), (5.5), and (5.7) has its own region of convergence (Fig. 4) determined by the inequalities (5.3), (5.4), and (5.6). Outside these regions of convergence, the solution must be taken in the form (2.6) and (2.7). The asymptotic expressions for the first terms u_2^P and u_2^s in (2.6) and (2.7) lead to series which are analogous to (5.1), (5.5), and (5.7), but which converge well everywhere

(5.8)

$$u_2^p = L\left(\frac{a}{r}\right)^{1/2}\left(1-\frac{a^2}{r^2}\right)^{1/4}\sum_k \frac{\exp\,[i\lambda_k(\arccos\varepsilon-\arccos a/r+{}^1/_2\,\pi)]}{w(t_k)(t_k-q_1^2)}\,\frac{\sin\lambda_k(\pi-\vartheta)}{\sin\lambda_k\pi}$$

$$u_2^p = iL\left(\frac{a}{r}\right)^{3/2}\left(1-\frac{a^2}{r^2}\right)^{-1/4}\sum_k \frac{\exp\,[i\lambda_k(\arccos\varepsilon-\arccos a/r+{}^1/_2\,\pi)]}{w(t_k)(t_k-q_1^2)}\,\frac{\cos\lambda_k(\pi-\vartheta)}{\sin\lambda_k\pi}$$

$$u_{21}^s = -\,iM\left(\frac{a}{r}\right)^{3/2}\left(1-\frac{a^2}{r^2}\right)^{-1/4}\sum_k \frac{\exp\,[i\lambda_k(2\arccos\varepsilon-\arccos\varepsilon a/r+{}^1/_2\,\pi)]}{t_k-q_1^2}\times$$
$$\times\,\frac{\sin\lambda_k(\pi-\vartheta)}{\sin\lambda_k\pi} \qquad (5.9)$$

$$u_{21}^s = -\,M\left(\frac{a}{r}\right)^{1/2}\left(1-\frac{a^2}{r^2}\right)^{1/4}\sum_k \frac{\exp\,[i\lambda_k(2\arccos\varepsilon-\arccos\varepsilon a/r+{}^1/_2\,\pi)]}{t_k-q_1^2}\times$$
$$\times\,\frac{\cos\lambda_k(\pi-\vartheta)}{\sin\lambda_k\pi}$$

$$u_{22}^s = -\,iN\,(k_2,r,a)\left(\frac{a}{r}\right)^{3/2}\left(1-\frac{a^2}{r^2}\right)^{-1/4}\sum_k \frac{\exp\,[-\,i\mu_k(\arccos a/r-{}^1/_2\,\pi)]}{w^2(\tau_k)(\tau_k-q_2^2)}\times$$
$$\times\,\frac{\sin\mu_k(\pi-\vartheta)}{\sin\mu_k\pi} \qquad (5.10)$$

$$u_{22}^s = -\,N\,(k_2,r,a)\left(\frac{a}{r}\right)^{1/2}\left(1-\frac{a^2}{r^2}\right)^{1/4}\sum_k \frac{\exp\,[-\,i\mu_k(\arccos a/r-{}^1/_2\,\pi)]}{w^2(\tau_k)(\tau_k-q_2^2)}\times$$
$$\times\,\frac{\cos\mu_k(\pi-\vartheta)}{\sin\mu_k\pi}$$

The series (5.1), (5.5), (5.7)-(5.10) represent the diffracted components of the displacements in the longitudinal and transverse waves. Thus, the field of longitudinal displacements is represented by the series (5.1) in the regions *1* and *2* (Fig. 4) and by the sum of the geometric terms (4.9) and the series (5.8) in the regions *3* and *4*. The transverse displacement components are represented by the series (5.5) and (5.7). However, as a consequence of the fact that the regions of rapid convergence of these series are different, the total solution can actually be used only in the region *1*, the region where both converge. Outside of region *1*, it is necessary to use solutions in the form of the sum of the series (5.9) and (5.10) and the integrals (4.2). The integrals (4.2) provide the geometrical part of the transverse displacement field (4.10) as long as $\nu_0 < x$; this corresponds to the region *4* of the physical space (Fig. 4). When $\nu_0 > x$, roots λ_k of the equation $\Delta_\nu = 0$ near the point x will be located between the path of steepest descent and EF (Fig. 6). Therefore, the integrals of (4.2) provide two terms: the geometrical part plus a sum of residues of the integrands at the zeros λ_k. This latter sum, together with the series (5.9), gives the series (5.5), which is rapidly convergent everywhere outside the region *4*. Thus, in regions *2* and *3*, the transverse displacement field will consist of the initial field, the reflected field, and the diffracted field represented by the series (5.5) and (5.10).

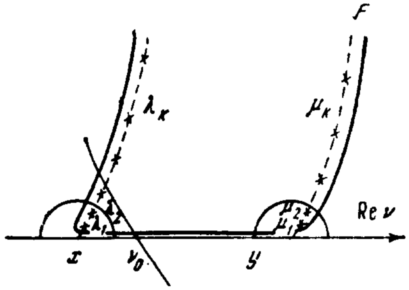

FIG. 6

Summing up, and omitting the surface wave components of the displacements, we may represent the entire displacement field in each region in the following form:

Region 1 $u = u_1^p + u_{11}^s + u_{12}^s,$
$v = v_1^p + v_{11}^s + v_{12}^s$

Region 2 $u = u_1^p + u_{11}^s + u_{22}^s + U^s,$
$v = v_1^p + v_{11}^s + v_{22}^s + V^s$

Region 3 $u = u_2^p + U^p + u_{11}^s + u_{22}^s + U^s$
$v = v_2^p + V^p + v_{11}^s + v_{22}^s + V^s$

Region 4 $u = u_2^p + U^p + u_{21}^s + u_{22}^s + U^s$
$v = v_2^p + V^p + v_{21}^s + v_{22}^s + V^s$

6. Analysis of the displacements in the transition regions. The equations for the longitudinal components of the displacement field (4.9) and (5.1) cease to be valid in the region ω_1 near the straight line (4.5), and those for the transverse displacements (4.10), (5.5), and (5.7) in the regions ω_2 and ω_3, near the straight lines (4.7) and (4.8) (Fig. 4). In the physical space, ω_1 corresponds to the region of the penumbra for the longitudinal components of displacement, ω_3 is the region of the penumbra for the transverse component, and ω_2 is the transition region near the angle of total internal reflection. In the ν-plane, these regions correspond to Fok regions for the Bessel functions. We shall now show to compute the displacement field in these regions.

In the region ω_1, the longitudinal displacements must be taken in the form of the sum of two terms. One of these is the series (5.8) which converge rapidly everywhere, and the other is the integral on EF in (2.6), for the computation of which the Hankel-Fok asymptotic expression for $H_\nu^{(1)}(x)$ (3.2) will be used. Then

$$U^p = \frac{ik_1}{2\varepsilon\sqrt{\pi\varepsilon}}\left(\frac{a}{r}\right)^{1/2}\int_E^F \frac{2\varepsilon^2p^2(t)-1}{p(t)}\,\frac{V(t,r,1)}{V(t,a,\varepsilon)}\,\frac{\exp(i\psi_1)\,dt}{w'(t)-q(t)\,w(t)} \tag{6.1}$$

where

$$\nu = x + (1/2\,x)^{1/3}t \equiv xp(t)$$

$$V(t,r,\varepsilon) = \left(1 - \frac{\varepsilon^2a^2p(t)}{r^2}\right)^{1/4}, \qquad q(t) = i\,\frac{[2\varepsilon^2p^2(t)-1]^2\,(1/2\,x)^{1/3}}{4\varepsilon^3p^2(t)\sqrt{1-\varepsilon^2p^2(t)}}$$

$$\psi_1 = xp(t)\,[1/2\,\pi - \vartheta - \arccos(ap(t)/r) + \arccos\varepsilon p(t)] +$$
$$+ k_1rV^2(t,r,1) - yV^2(t,a,\varepsilon)$$

The quantity z_1, which is defined in (5.3) is $\sim \cos\beta_1$ near the boundary of the shadow. Therefore, in the region ω_1, the relation $|z_1| \sim x^{-1/3}$ is valid. In the illuminated region for the longitudinal displacements, *3*, *4*, (Fig. 4), $z_1 x^{1/3} \ll -1$, and in the region of the full shadow, *1*, *2* (Fig. 4), $z_1 x^{1/3} \gg 1$. In the penumbra, where $|z_1| x^{1/3} \sim 1$, the integrals (6.1) must be computed by a quadrature method. To do this, it is convenient to transform the integration path into the broken line Γ going from $\infty \exp(2\pi i/3)$ to 0 along the straight line $\operatorname{arc} t = {}^2/_3\pi$ and from 0 to ∞ along the real axis and then to use the representation of the function w on the complex ray $\operatorname{arc} t = 2\pi/3$ [6]. The transformed integrals are calculated from tables of the functions $u(t)$ and $v(t)$ and their derivatives which are given in [6].

When $z_1 x^{1/3} \ll -1$, the principal part of the integration in the integral of (6.1) will lie, after the proper deformation of the path, near the large values of t. For these t, the asymptotic formula

$$w(t) = (-t)^{-1/4} \exp\left[{}^2/_3 i (-t)^{3/2} + {}^1/_4 \pi i\right]$$

of [6] for the Airy function is applicable. Substituting these values into (6.1), we compute the integrals by the stationary phase method and obtain a result which for angles of incidence near $\alpha^\times$ coincides with the expressions for the 'geometrical' components of the displacements.

When $z_1 x^{1/3} \gg 1$, the integrals of (6.1) can be calculated by means of the residues of the integrand at the zeros of (3.4). We then obtain series which converge well in the region of the shadow and which, when summed with the series (5.8), agree with the solution (5.1) obtained earlier for the region of the shadow. In this way, the solution in the region of the penumbra ties together, as it were, the solutions obtained for the illuminated region and for the region of the shadow.

The expressions for the components of the transverse displacements cease to be valid in the regions ω_2 and ω_3. For the region ω_2, we take the solution (2.7), replacing the series in ν_k by their asymptotic representations – the series (5.9) and (5.10); to compute the second integral along EF, the Hankel-Fok asymptotic expressions for $H_\nu^{(1)}(x)$ must be used. Then

$$U^s = -\frac{i}{2r}\int_E^F \nu \frac{H_\nu^{(2)}(y)}{H_\nu^{(1)}(y)} H_\nu^{(1)}(k_2 r)\, e^{i\nu({}^1/_2\pi - \vartheta)}\, d\nu - \tag{6.2}$$
$$- ik_1 \left(\frac{a}{r}\right)^{1/2}\left(\frac{x}{2}\right)^{1/3}\sqrt{\frac{2}{\pi k_2 a}}\, e^{i\pi/4}\int_E^F \frac{p(t)\, w'(t) \exp(i\psi_2)\, dt}{V(t, r, \varepsilon)\,[w'(t) - q(t)\, w(t)]}$$

where

$$\psi_2 = k_2 r V^2(t, r, \varepsilon) - 2yV^2(t, a, \varepsilon) + xp(t)\,[{}^1/_2\pi - \vartheta -$$
$$- \operatorname{arc}\cos \varepsilon a p(t)/r + 2 \operatorname{arc}\cos \varepsilon p(t)]$$

In the transition region near the angle of total internal reflection $|z_2| \sim y^{-2/3}$, the second integral in (6.2) must again be performed by numerical quadrature, after transforming the path of integration into the path Γ.

In region *4* (Fig. 4) $z_2 y^{2/3} \ll -1$, and the principal part of the integration in the integral (6.2) for the method of stationary phase will lie near large negative values of t. The integrals (6.2) then give a solution which coincides on some part of the region *4* with the solution which was obtained before, (4.10). For the region $z_2 y^{2/3} \gg 1$ and $x < x + ({}^1/_2 x)^{1/3} t < y$ the value of t will be large and positive. Therefore, the asymptotic expression [6]

$$u(t) = t^{-1/4} \exp ({}^2/_3 t^{3/2}), \qquad v(t) = {}^1/_2 t^{-1/4} \exp (-{}^2/_3 t^{-3/2})$$

should be used for $w(t)$.

Moreover, the fact that poles of the integrand of (6.2) lie between the line of steepest descent and the path of integration should be considered. Then, in some interval of integration $t > 0$, lying outside the Fok region $\nu \sim x$, u^s in (6.2) will consist of two parts; the first, computed as an integral near a saddle point, coincides with the 'geometric' components of the transverse displacements; the second is a series in residues at t_k which, when summed with the series (5.9) results in the series (5.5) for u_{11}^s, this latter series converging well for $z_2 x^{1/3} \gg 1$. Thus, Eq. (6.2) ties together the solutions in the regions *3* and *4*.

In the region ω_3, we once more take the solution in the form (2.7), replace the series in ν_k by their asymptotic representations (5.9) and (5.10), and make use of the Hankel-Fok asymptotic representation of $H_\nu^{(1)}(y)$ to compute the integral along EF. In the transformation of the contour EF of the ν-plane into the corresponding contour Γ in the τ-plane, we note that the poles $x + ({}^1/_2 x)^{1/3} t_k$ of the integrand will lie between the contours. Therefore, the integral along EF is replaced by an integral along Γ in the τ-plane plus a series in the residues of the integrands at $x + ({}^1/_2 x)^{1/3} t_k$. This last series when summed with (5.9) again gives the series (5.5). Thus, in the region ω_3, the solution is represented by the series (5.5) and (5.10) and integrals along the path Γ in the τ-plane. Just as in (6.1), these integrals must be calculated by a numerical quadrature method. It is easy to show that for $z_3 y^{1/3} \ll -1$, i.e., in the illuminated region for the transverse displacements, the integrals give a geometric displacement field which coincides with (4.10) in some interval.

For $z_3 y^{1/3} \gg 1$, i.e., in the region of full shadow, the integrals are computed by means of residues of the integrands and result in a series which when summed with (5.10) coincides with the series (5.7). Thus, the solutions in the regions ω_1, ω_2, and ω_3 tie together the solutions obtained earlier in Sections 4 and 5.

7. Diffraction of a longitudinal wave. With the same formulation as in Section 1, it is possible to consider a longitudinal elastic wave with the potential

$$\varphi_0 = \exp (ik_1 r \cos \vartheta - i\omega t).$$

incident on a cylinder. The complete solution of this problem on the surface of the cylinder is given in [15]; here we shall give expressions for the displacements in the diffracted waves because we shall require them in Section 8:

$$u_3^p = N(k_1, r, a)\left(\frac{a}{r}\right)^{1/2}\left(1-\frac{a^2}{r^2}\right)^{1/4}\sum_k \frac{\exp\left[-i\lambda_k\left(\arccos a/r + {}^1/_2\pi\right)\right]}{w^2(t_k)(t_k - q_1{}^2)}\,\frac{\cos\lambda_k\vartheta}{\sin\lambda_k\pi}$$

$$v_3^p = iN(k_1, r, a)\left(\frac{a}{r}\right)^{3/2}\left(1-\frac{a^2}{r^2}\right)^{-1/4}\sum_k \frac{\exp\left[-i\lambda_k\left(\arccos a/r + {}^1/_2\pi\right)\right]}{w^2(t_k)(t_k - q_1{}^2)}\,\frac{\sin\lambda_k\vartheta}{\sin\lambda_k\pi} \tag{7.1}$$

and for the transverse components

$$u_3^s = -iP\left(\frac{a}{r}\right)^{3/2}\left(1-\varepsilon^2\frac{a^2}{r^2}\right)^{-1/4}\sum_k \frac{\exp\left[i\lambda_k\left(\arccos\varepsilon - \arccos\varepsilon a/r - {}^1/_2\pi\right)\right]}{w(t_k)(t_k - q_1{}^2)}\,\frac{\cos\lambda_k\vartheta}{\sin\lambda_k\pi}$$

$$u_3^s = -P\left(\frac{a}{r}\right)^{1/2}\left(1-\varepsilon^2\frac{a^2}{r^2}\right)^{1/4}\sum_k \frac{\exp\left[i\lambda_k\left(\arccos\varepsilon - \arccos\varepsilon a/r - {}^1/_2\pi\right)\right]}{w(t_k)(t_k - q_1{}^2)}\,\frac{\sin\lambda_k\vartheta}{\sin\lambda_k\pi} \tag{7.2}$$

$$P \equiv P(\varepsilon, k_2, a, r) = k_2(2\varepsilon^2 - 1)\,\pi^{1/2}(1-\varepsilon^2)^{-1/4}\exp\left[ik_2\left(\sqrt{r^2 - a^2\varepsilon^2} - a\sqrt{1-\varepsilon^2}\right)\right]$$

The region of rapid convergence of (7.1) is determined by the condition

$$-\arccos(a/r) - \vartheta + {}^1/_2\pi > 0$$

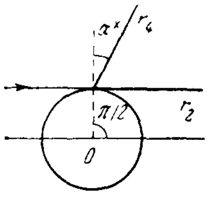

FIG. 7

i.e., the region of the geometric shadow (Fig. 7). For the series of (7.2), the region of rapid convergence lies to the right of the ray whose equation is $r_4 = a\varepsilon / \cos(\vartheta^\times - \vartheta)$, i.e.,

$$\arccos\varepsilon - \arccos(\varepsilon a/r) - \vartheta + {}^1/_2\pi > 0$$

Outside of their regions of convergence the longitudinal and transverse components of displacement are represented by the sum of geometrical and diffraction terms. The latter are series analogous to (7.1) and (7.2) replacing the factors $\sin\lambda_k\theta$ and $\cos\lambda_k\theta$, in accordance with (2.5), by $-\sin\lambda_k(\pi - \vartheta)\exp(i\lambda_k\pi)$ and $\cos\lambda_k(\pi - \vartheta)\exp(i\lambda_k\pi)$; these series converge rapidly everywhere.

8. The geometrical theory of diffraction of plane elastic waves by an arbitrary convex cylinder. We shall use Keller's method in the geometric theory of diffraction, with some modifications, in order to solve problems of the diffraction of elastic waves. We shall pose the problem as follows: A harmonic wave, longitudinal or transverse, is incident on a smooth, convex cylinder in a homogeneous elastic medium. It is required to find the diffracted displacement field outside the diffracting body. In contrast to [11, 12], we shall consider that diffracted rays are produced not only by the initial longitudinal and transverse rays which are tangent to the surface of the body, but also by the initial transverse rays which are incident at the angle of total internal reflection $\alpha^{\times}$. Moreover, a surface longitudinal diffracted ray will emit, not only longitudinal diffracted rays, but transverse diffracted rays also, these making an angle $\alpha^{\times}$ with the normal to the surface of the body.

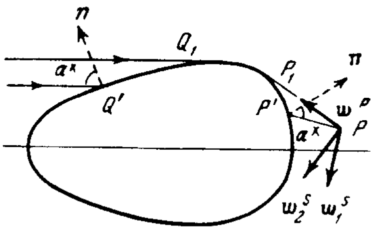

FIG. 8

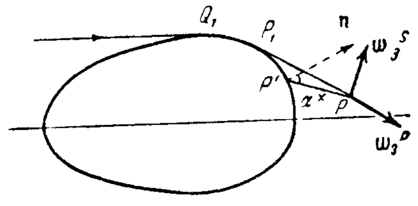

FIG. 9

All these diffracted rays will correspond to the extended Fermat principle, i.e., they contain geodesic arcs along the surface of the body and straight lines from the body to the point P under consideration (Figs. 8 and 9). We shall characterize the diffracted field w on each ray by the amplitude A, which is a vector for a vector field, and the phase δ.

In the case of an incident transverse wave, the diffracted field $\mathrm{w}_d(P)$ at the point P will be formed from the sum of the fields on rays of three types: a longitudinal and a transverse diffracted ray emitted at the point P_1 (Fig. 8), and a transverse diffracted ray of the head-wave type emitted at P'. Let the initial ray at the point Q_i be characterized by the amplitude $\mathbf{A}_0(Q_i)$ and the phase $\delta(Q_i)$. The displacement field at the point P which is associated with the three diffracted rays beginning at Q_1 and Q' will be

$$\mathrm{w}_d(P) = \mathrm{w}^p(P) + \mathrm{w}_1^s(P) + \mathrm{w}_2^s(P) \tag{8.1}$$

where w^p and w^s are the displacements on the longitudinal and transverse rays P_1P and w_1^s that on the transverse ray $P'P$. It will be assumed that the direction associated with the displacement amplitude remains constant along a straight ray and that its magnitude is determined by the same construction as is given in [12] for a scalar field. Then

$$\begin{aligned} w^p(P) = {} & A_0(Q')\exp\{ik_2\delta_0(Q') + ik_1(\sigma_1 + s_1)\}(s_1)_{P_1}^{-1/2}\times \\ & \times\sum_k D_{k,p}^{\ s}(Q')\,D_{k,p}^{\ p}(P_1)\exp\left\{-\int_0^{\sigma_1}\alpha_k(\sigma)\,d\sigma\right\} \end{aligned} \tag{8.2}$$

$$\begin{aligned} w_1^s(P) = {} & A_0(Q')\exp\{ik_2[\delta_0(Q') + s_2] + ik_1\sigma_2\}[\rho/(\rho + s_2)]_{P'}^{1/2}\times \\ & \times\sum_k D_{k,p}^{\ s}(Q')\,D_{k,s}^{\ p}(P')\exp\left\{-\int_0^{\sigma_2}\beta_k(\sigma)\,d\sigma\right\} \end{aligned} \tag{8.3}$$

$$w_2^s(P) = A_0(Q_1)\exp\{ik_2[\delta_0(Q_1) + s_1 + \sigma_3]\}(s_1)_{P_1}^{-1/2} \times \times \sum_k D_{k,s}^{s}(Q_1)\, D_{k,s}^{s}(P_1)\exp\left\{-\int_0^{\sigma_3}\gamma_k(\sigma)\,d\sigma\right\} \tag{8.4}$$

where σ_i is a distance along a geodesic arc on the surface: $\sigma_1 = Q'P_1$, $\sigma_2 = Q'P'$, $\sigma_3 = Q_1P_1$, s_i is a distance along a straight line: $s_1 = P_1P$ and $s_2 = P'P$, ρ is the curvature of the wave front, and $D(Q_i)$ are the diffraction coefficients at the point Q_i, which are equal to ratios of the amplitudes on the various rays at this point. The superscript on the coefficient D denotes which type of ray, longitudinal (p) or transverse (s) is transformed into another, the latter being denoted by the subscript. Thus, $D_{k,p}^{s}(Q_i)$ is the ratio of the amplitude of the transverse ray to the amplitude of the diffracted longitudinal ray at the point Q_i, etc; γ_k denotes the decay exponent of a transverse surface ray as a result of emission of transverse rays, and α_k and β_k are decay exponents of a longitudinal surface ray as a consequence of emission of longitudinal and transverse diffracted rays. The subscript k indicates that the field on the diffracted field on the surface diffracted wave consists of a set of different frequencies. Each frequency is characterized by its own diffraction coefficient and decay exponents, either α_k, β_k, or γ_k. On the basis of the principle of reciprocity, it will be assumed that diffraction coefficients with identical upper and lower scripts are the same functions of their variables.

Knowing the form of the body and the coordinates of the point P, one can always find the direction of the displacement (Figs. 8 and 8); if the sign of the displacement is taken incorrectly, this is taken into account when determining the corresponding diffraction coefficient by the sign of the coefficient.

In the case of a longitudinal wave incident on the body, a diffracted longitudinal surface ray is formed at the point of tangency. This gives rise to two families of diffracted longitudinal and transverse rays. The diffracted field at the point P (Fig. 9) will equal

$$\mathbf{w}_d(P) = \mathbf{w}_3^p(P) + \mathbf{w}_3^s(P) \tag{8.5}$$

$$w_3^p = A_0(Q_1)\exp\{ik_1[\delta_0(Q_1) + \sigma_3 + s_1)\}(s_1)_{P_1}^{-1/2} \times \times \sum_k D_{k,p}^{p}(Q_1)\, D_{k,p}^{p}(P_1)\exp\left\{-\int_0^{\sigma_3}\alpha_k(\sigma)\,d\sigma\right\} \tag{8.6}$$

$$w_3^s = A_0(Q_1)\exp\{ik_1[\delta_0(Q_1) + \sigma_4] + ik_2s_2\}\left(\frac{\rho}{\rho + s_2}\right)_{P'}^{1/2} \times \times \sum_k D_{k,p}^{p}(Q_1)\, D_{k,s}^{p}(P')\exp\left\{-\int_0^{\sigma_4}\beta_k(\sigma)\,d\sigma\right\} \tag{8.7}$$

where σ_4 is the geodesic arc between Q_1 and P' (Fig. 9).

Further, following Keller, we shall consider that the diffraction coefficients $D(Q_i)$ and the decay exponents depend to a first approximation on the character of the fields k_1 and k_2 and the properties of the body at the point Q_i (more precisely, on the radius of curvature of the body at the point Q_i). Therefore, the diffraction coefficients and the decay exponents may be determined by solving the problem of diffraction of longitudinal and transverse waves by any body of simple shape.

Thus, in order to solve the problem of diffraction of plane elastic waves by a cylindrical body of arbitrary section, it will suffice to find the solution for the diffraction of plane elastic waves by a circular cylinder of radius a with the corresponding boundary conditions and to carry out the asymptotic expansion of the solution in $1/k_1$ and $1/k_2$. Next, the asymptotic expansion and the solution by Keller's method are compared and diffraction coefficients and decay exponents are found as functions of k_1, k_2, a. With the shape of the body known, the solution can then be constructed with the aid of the coefficients which have been found.

We shall apply the preceding method to find the displacements in an elastic medium when plane elastic waves are diffracted by a circular cylinder. We make the following comment: since the cylinder has constant radius of curvature and the elastic medium is homogeneous, the decay exponents are constant along a ray and depend only on k_1, k_2, and a.

Let us now examine the problem of the incidence of a transverse wave on a circular cylinder. Three types of diffracted rays will arrive at a point P which is in the geometric shadow: a longitudinal ray (8.2), a transverse one (8.4), and a transverse one of the head-wave type (8.3). Besides the three rays which arrive at P from the top of the cylinder, it is also necessary to consider the three rays from the bottom of the cylinder and all those rays which go through the point P after having encircled the cylinder n times.

Summing up all the fields on these rays and taking account of the orientation of the displacements, we find the field on the transverse rays at point P (Fig. 10)

$$\begin{aligned} u_2^s(P) = &- ik_2\left(\frac{a}{r}\right)^{3/2}\left(1-\frac{a^2}{r^2}\right)^{-1/4} a^{-1/2} e^{ik_2\sqrt{r^2-a^2}} \sum_k (D_{k,s}^s)^2 \times \\ &\times \exp\left[-\left(\frac{\pi}{2}+\arccos\frac{a}{r}\right)(ik_2-\gamma_k)\,a\right]\frac{\sin\vartheta a\,(ik_2-\gamma_k)}{\sin\pi a\,(ik_2-\gamma_k)} \end{aligned} \tag{8.8}$$

Here and in what follows, formulas are given only for the radial component u of the displacements in the diffracted field. In Figs. 10-14 the directions we have taken for the displacements on the upper diffracted rays (P_1P, $P'P$) and the lower ones (P_2P, $P''P$) are denoted by arrows at the point P. The transverse rays which strike the cylinder at the angle $\alpha^\times$ give rise to longitudinal and transverse diffracted rays.

The displacement component on the longitudinal rays (Fig. 11) are

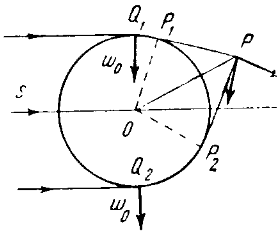

FIG. 10

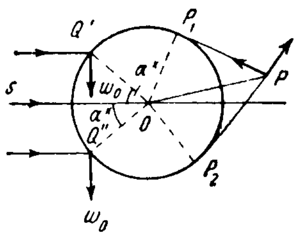

FIG. 11

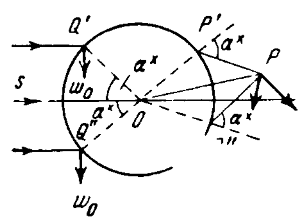

FIG. 12

$$u^p(P) = -ik_2\left(\frac{a}{r}\right)^{1/2}\left(1-\frac{a^2}{r^2}\right)^{1/4} a^{-1/2} e^{i\left(k_1\sqrt{r^2-a^2}-k_2 a\sqrt{1-\varepsilon^2}\right)}\sum_k D_{k,p}^{s} D_{k,p}^{p}$$
$$\times \exp\left[\left(\arc\cos\varepsilon - \arc\cos\frac{a}{r}-\frac{\pi}{2}\right)(ik_1-\alpha_k)\,a\right]\frac{\sin\vartheta a\,(ik_1-\alpha_k)}{\sin\pi a\,(ik_1-\alpha_k)} \qquad (8.9)$$

and for the transverse displacements of the head-wave type (Fig. 12)

$$u_1^s(P) = -ik_2\,\frac{\varepsilon(1-\varepsilon^2)^{1/4}}{(1-\varepsilon^2a^2/r^2)^{1/4}}\left(\frac{a}{r}\right)^{1/2} e^{ik_2\left(\sqrt{r^2-\varepsilon^2a^2}-2a\sqrt{1-\varepsilon^2}\right)}\times$$
$$\times\sum_k D_{k,p}^{s} D_{k,s}^{p}\exp\left[\left(2\arc\cos\varepsilon-\arc\cos\frac{\varepsilon a}{r}-\frac{\pi}{2}\right)(ik_1-\beta_k)\,a\right]\frac{\sin\vartheta a\,(ik_1-\beta_k)}{\sin\pi a\,(ik_1-\beta_k)} \qquad (8.10)$$

By comparing the results obtained for the displacements (8.8)-(8.10) with the asymptotic expressions for the region of the geometric shadow in the exact method (5.7)-(5.9), we can obtain only products of the diffraction coefficients. This, however, is not sufficient. To solve the problem for a body of arbitrary shape by Keller's method it is necessary to know the values of these coefficients individually as functions of k_1, k_2, and the radius of curvature of the body.

We shall therefore examine another case, that of incidence of a plane longitudinal wave on the cylinder, with the same boundary conditions. We shall obtain the field at the point P as the sum of fields on all the diffracted longitudinal rays (8.6) and transverse rays (8.7) which pass through P.

The displacement components in the longitudinal diffracted waves are (Fig. 13)

$$u_3^p = -k_1\left(\frac{a}{r}\right)^{1/2}\left(1-\frac{a^2}{r^2}\right)^{1/4} a^{-1/2} e^{ik_1\sqrt{r^2-a^2}}\sum_k (D_{k,p}^{p})^2\times$$
$$\times\exp\left[-\left(\arc\cos\frac{a}{r}+\frac{\pi}{2}\right)(ik_1-\alpha_k)\,a\right]\frac{\cos\vartheta a\,(ik_1-\alpha_k)}{\sin\pi a\,(ik_1-\alpha_k)} \qquad (8.11)$$

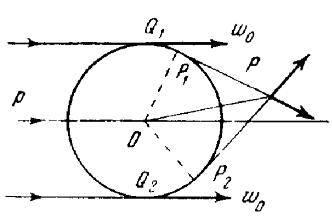

FIG. 13

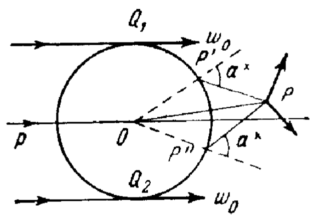

FIG. 14

The field for the transverse diffracted waves is found in a similar fashion (Fig. 14)

$$u_3^s = -k_1\varepsilon\,(1-\varepsilon^2)^{1/4}\left(1-\varepsilon^2\frac{a^2}{r^2}\right)^{-1/4}\left(\frac{a}{r}\right)^{3/2} e^{ik_2(\sqrt{r^2-a^2\varepsilon^2}-a\sqrt{1-\varepsilon^2})}\times \tag{8.12}$$
$$\times\sum_k D_{k,p}^{p} D_{k,s}^{p}\exp\left[\left(\arccos\varepsilon-\arccos\frac{\varepsilon a}{r}-\frac{\pi}{2}\right)(ik_1-\beta_k)\,a\right]\frac{\cos\vartheta a\,(ik_1-\beta_k)}{\sin\pi a\,(ik_1-\beta_k)}$$

Comparing (8.11) and (8.12) with the asymptotic expressions of the exact solution for the region of the geometric shadow, (7.1) and (7.2) and the analogous expressions for the diffraction of a transverse wave, we obtain

$$\begin{aligned}\alpha_k = \beta_k = ik_1 - i\lambda_k a^{-1} &= -\,ia^{-1}(k_1a/2)^{1/3}t_k\\ \gamma_k = ik_2 - i\mu_k a^{-1} &= -\,ia^{-1}(k_2a/2)^{1/3}\tau_k\end{aligned} \tag{8.13}$$

where t_k and τ_k are the roots of Eqs. (3.4) and (3.5), and also five equations for four diffraction coefficients. From these, we find

$$D_{k,s}^{s} = \left(\frac{8\pi}{k_2}\right)^{1/4}\frac{ie^{-i\pi/8}}{w(\tau_k)\,(\tau_k-q_2^2)^{1/2}}\left(\frac{k_2a}{2}\right)^{1/6},\qquad D_{k,p}^{p} = \left(\frac{8\pi}{k_1}\right)^{1/4}\frac{ie^{-i\pi/8}}{w(t_k)\,(t_k-q_1^2)^{1/2}}\left(\frac{k_1a}{a}\right)^{1/6}$$
$$D_{k,s}^{p} = \left(\frac{k_1\pi}{8}\right)^{1/4}\frac{2\varepsilon^2-1}{\varepsilon^2(1-\varepsilon^2)^{1/2}}\frac{e^{i\pi/8}}{(t_k-q_1^2)^{1/2}}\left(\frac{2}{k_1a}\right)^{1/6} \tag{8.14}$$
$$D_{k,p}^{s} = -\left(\frac{k_1\pi a^2}{8}\right)^{1/4}\frac{2\varepsilon^2-1}{\varepsilon^{1/2}(1-\varepsilon^2)^{1/4}}\frac{e^{i\pi/8}}{(t_k-q_1^2)^{1/2}}\left(\frac{2}{k_1a}\right)^{1/6}$$

We note that the surface of the diffracting body is a caustic of the diffracted rays (except the transverse diffracted rays of the head-wave type), and that the displacements in the Keller method will, therefore, become infinite on the surface. In order to determine the surface displacements, we turn to the asymptotic expression of the exact solution. Near the surface as $r \to a$ we use the Hankel-Fok asymptotic formulas for the Hankel functions $H_\nu^{(1)}(k_1r)$ and $H_\nu^{(1)}(k_2r)$ and their derivatives instead of the Debye formulas. Therefore, the expressions for displacements near and on the body which are obtained by the Keller method must be multiplied by the ratio of the Hankel-Fok asymptotic representation of the proper function to the Debye asymptotic expression.

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BIBLIOGRAPHY

1. Kupradze, V.D., *Osnovnye zadachi matematicheskoi teorii difraktsiia (Basic Problems of the Mathematical Theory of Diffraction)*, ONTI, 1934.

2. Skuridin, G.A., K teorii rasseianiia uprugikh voln na krivolineinoi granitse (Theory of the scattering of elastic waves by a curved boundary), *Izv. AN SSSR, Ser. geofiz.*, 1957, No. 2.

3. Skuridin, G.A., Priblizhennaia teoriia golovnoi volny, voznikaiushchei na tsilindricheskom vkliuchenii v odnorodnoi uprugoi srede (An approximate theory of the head wave generated by a cylindrical inclusion in a homogeneous elastic medium), *PMM*, 1961, Vol. 25, No. 3.

4. Gilbert, F., and Knopoff, L., Scattering of impulsive elastic waves by a rigid cylinder, *J. Acoust. Soc. America*, 1959, Vol. 31, No. 9.

5. Watson, G.N., The diffraction of electric waves by the Earth, *Proc. Roy. Soc. A*, 1918, Vol. 95, No. 666.

6. Fok, V.A., *Difraktsiia radiovoln vokrug zemnoi poverkhnosti (Diffraction of Radio Waves Around the Earth's Surface)*, Izd-vo AN SSSR, 1946.

7. Bremmer, H., *Terrestrial Radio Waves*, Elsevier Pub. Co., New York, Amsterdam, London, Brussels, 1949.

8. Franz, W. Über die Greenschen Funktionen des Zylinders und der Kugel, *Z. Naturforsch.*, 1954, B. 9a, No.9.

9. Imai, I., Die Beugung elektromagnetischer Wellen an einem Kreiszylinder, *Z. Phys.*, 1954, B. 137, No. 1.

10. Gorianov, A.S., Asimptoticheskoe reshenie zadachi o difraktsii ploskoi elektromagnitnoi volny na provodiashchem tsilindre (An asymptotic solution of the problem of diffraction of a plane electromagnetic wave by a conducting cylinder), *Radiotekhnika i Elektronika*, 1958, Vol. 3, No. 5.

11. Keller, J.B., Lewis, R. and Secler, B.D., Asymptotic solution of some diffraction problems, *Comm. Pure and Appl. Math.*, 1956, Vol. 9, No. 2.

12. Levy, B.R. and Keller, J.B., Diffraction by a smooth object, *Comm. Pure and Appl. Math.*, 1959, Vol. 12, No. 1.

13. Petrashem', G.I., Smirnova, N.S., and Makarov, G.I., Ob asimptoticheskikh predstavleniiakh tsilindricheskikh funktsii (On asymptotic representations of Bessel functions), *Uchenye zapiski, Leningrad State University*, 1953, vyp. 27, No. 170.

14. Hönl, H., Maue, A., and Westphal, K., *Teoriia difraktsii (The Theory of Diffraction)* (Russian translation) Izd. 'Mir', 1964.

15. Iavorskaia, I.M., Difraktsiia ploskoi prodol'noi volny na krugovom tsilindre (Diffraction of a plane longitudinal wave by a circular cylinder), *Dokl. AN SSSR*, 1964, Vol. 158, No. 6.

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